Calculus II - Day 19

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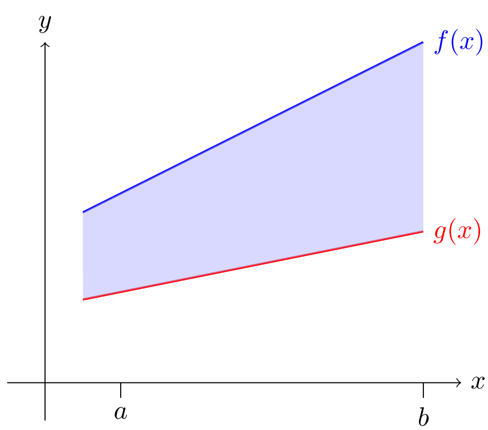
Goals for today:

- Find the volume of a region revolved around the y-axis without changing variable (*"shells" method*).
- Find the length of a curve using integration.

Last week: To find the volume of a solid obtained by revolving a region R around the x-axis:

$$V = \int_a^b \pi \left(f(x)^2 - g(x)^2 \right) \, dx$$



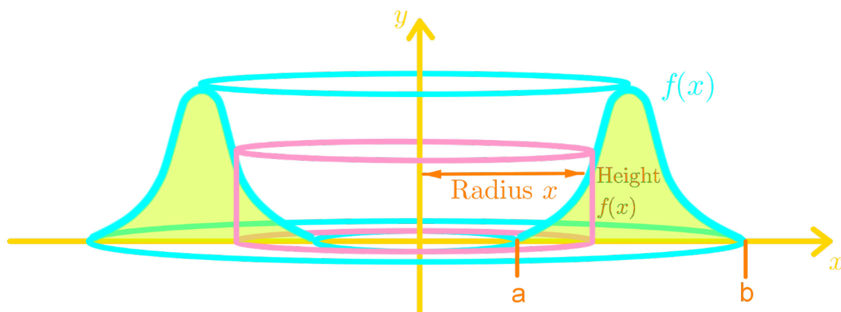
About the y-axis:

$$(y = f(x) \text{ becomes } x = f^{-1}(y) \text{ and } y = g(x) \text{ becomes } x = g^{-1}(y))$$

$$V = \int_c^d \pi \left(f^{-1}(y)^2 - g^{-1}(y)^2 \right) \, dy$$

We can rotate about the y-axis w/o changing variable by decomposing the solid into <u>cylindrical shells</u> instead of washers or disks.

$$\text{Volume of Solid} = \int_{\text{lower bound}}^{\text{upper bound}} 2\pi(\text{radius})(\text{height})dx$$



The volume of the solid is approximated by

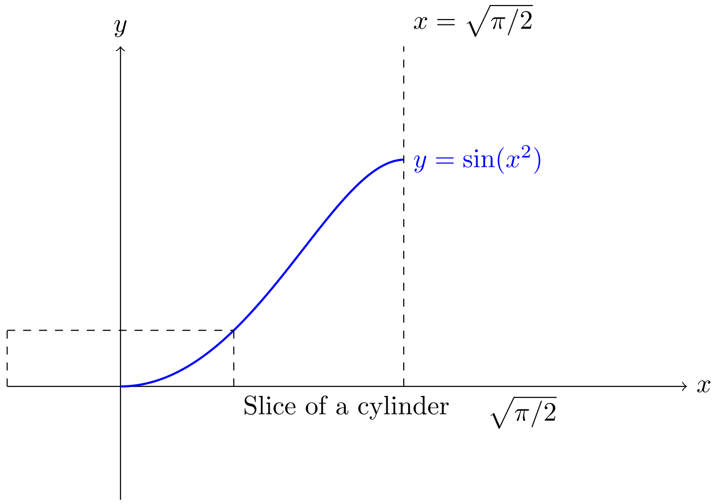
$$V \approx \sum_{k=1}^{n} 2\pi x_k f(x_k)\Delta x$$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi x_k f(x_k)\Delta x = \int_a^b 2\pi x f(x)\, dx$$

We can generalize if R is the region between f and g:

$$V = \int_a^b 2\pi x(f(x) - g(x))\, dx$$

Example (Sine bowl): Let R be the region bounded by the graphs of $f(x) = \sin(x^2)$, the x-axis, and the vertical line $x = \sqrt{\pi/2}$. Find the volume of the solid obtained by revolving R about the y-axis.



$$V = \int_0^{\sqrt{\pi/2}} 2\pi x \sin(x^2)\, dx$$

Let $u = x^2$, $du = 2x\, dx$,

$$u(0) = 0^2 = 0, \quad u(\sqrt{\pi/2}) = \pi/2$$

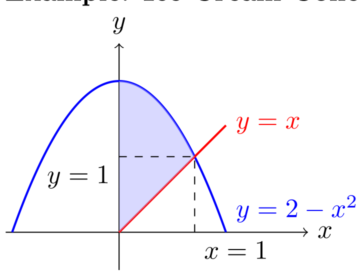
$$= \int_0^{\pi/2} \pi \sin(u)\, du$$

$$= -\pi \cos(u)\Big|_0^{\pi/2} = -\pi\,(0-1) = \boxed{\pi}$$

What if we used washers instead?
Outer radius: $x = \sqrt{\pi/2}$ Inner radius: $y = \sin(x^2) \Rightarrow x^2 = \arcsin(y) \Rightarrow x = \sqrt{\arcsin(y)}$

$$V = \int_0^1 \pi \left(\frac{\pi}{2} - \arcsin(y)\right) dy$$

... harder to integrate!
Example: Ice Cream Cone



Revolve R about the y-axis and find the volume:
1) **Using shells:**

$$V = \int_0^1 2\pi x \left(2 - x^2 - x\right)\, dx$$

2) **Using disks:**

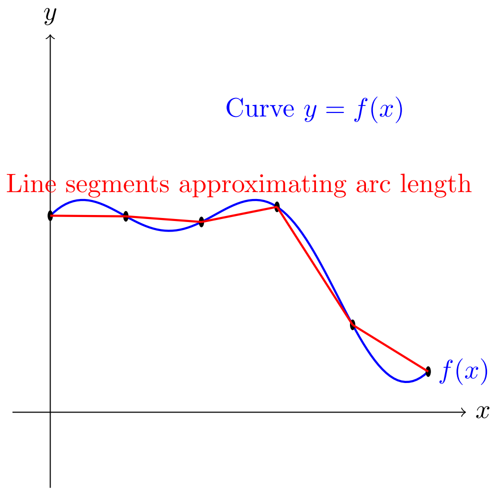
$$V = \int_1^2 \pi \left(\sqrt{2-y}\right)^2\, dy + \int_0^1 \pi y^2\, dy$$

Answer (either way): $\boxed{\dfrac{5\pi}{6}}$

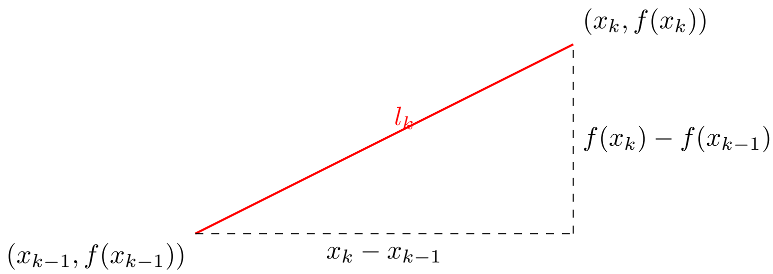
Arc length:
Let $f(x)$ be a function with a continuous derivative on $[a,b]$. The length of the curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Where does this come from?



Goal: Estimate the length of the curve by dividing $[a, b]$ into n parts and calculate the length of the line segments between consecutive endpoints.



$$|l_k| = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$$

Set $\Delta x = x_k - x_{k-1}$ for every k, $\Delta y_k = f(x_k) - f(x_{k-1})$,

$$|l_k| = \sqrt{(\Delta x)^2 + (\Delta y_k)^2}, \text{ so}$$

$$L \approx \sum_{k=1}^{n} |l_k| = \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$$

$$= \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k^2}{\Delta x^2}\right)} \Delta x$$

As $n \to \infty$:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example: Find the length of the arc of the "semicubical paraboloid" $x^{3/2}$ between $(1,1)$ and $(4,8)$.

$$f'(x) = \frac{3}{2}x^{1/2}, \quad (f'(x))^2 = \frac{9}{4}x$$

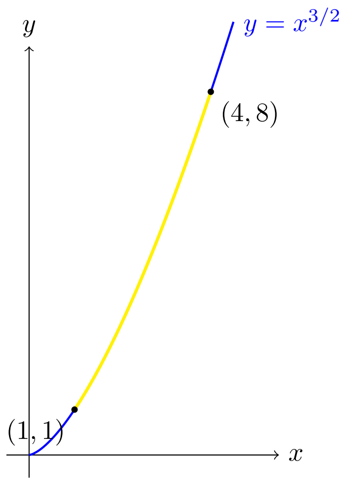
$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

Substitute: $u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}dx$

$$u(1) = 1 + \frac{9}{4}(1) = \frac{13}{4}, \quad u(4) = 1 + \frac{9}{4}(4) = 10$$

$$L = \int_{\frac{13}{4}}^{10} \frac{4}{9}\sqrt{u}\, du = \frac{4}{9} \cdot \frac{2}{3}u^{3/2}\Big|_{\frac{13}{4}}^{10}$$

$$L = \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right)$$



Example: Find the length of the curve $y = x^2$ from $(0,0)$ to $(2,4)$.

$$f'(x) = 2x, \quad (f'(x))^2 = 4x^2$$

$$L = \int_0^2 \sqrt{1 + 4x^2}\, dx = \int_0^2 2\sqrt{\frac{1}{4} + x^2}\, dx$$

Substitute $x = \frac{1}{2}\tan(\theta)$, $dx = \frac{1}{2}\sec^2(\theta)\, d\theta$.

When $x = 0$, what is θ?

$$0 = \frac{1}{2}\tan(\theta) \implies \theta = 0$$

When $x = 2$, what is θ?

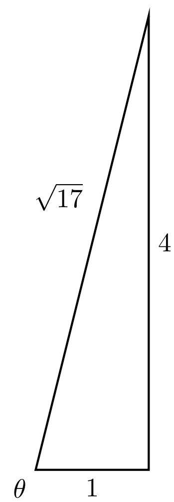
$$2 = \frac{1}{2}\tan(\theta) \implies \tan(\theta) = 4$$
$$\implies \theta = \arctan(4)$$

$$L = \int_0^{\arctan(4)} 2\sqrt{\frac{1}{4} + \frac{1}{4}\tan^2(\theta)} \cdot \frac{1}{2}\sec^2(\theta)\, d\theta$$

$$= \int_0^{\arctan(4)} \frac{1}{2}\sec^3(\theta)\, d\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta) + \tan(\theta)|\right)\Big|_0^{\arctan(4)}$$

Using a triangle for $\theta = \arctan(4)$:



$$L = \frac{1}{4}\sec(\arctan(4))\tan(\arctan(4)) + \frac{1}{4}\ln|\sec(\arctan(4)) + \tan(\arctan(4))|$$

From the triangle:

$$\sec(\arctan(4)) = \frac{\sqrt{17}}{1}, \quad \tan(\arctan(4)) = \frac{4}{1}$$

$$L = \frac{1}{4}(\sqrt{17} \cdot 4) + \frac{1}{4}\ln|\sqrt{17} + 4|$$

$$L = \sqrt{17} + \frac{1}{4}\ln|\sqrt{17} + 4|$$

$$\approx 4.647\ldots$$